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Noncommutative geometry from strings and branes

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Abstract: Noncommutative torus compactification of Matrix model is shown to be a direct consequence of quantization of the open strings attached to a D-membrane with a non-vanishing background B field. We calculate the BPS spectrum of such a brane system using both string theory results and DBI action. The DBI action leads to a new transformation property of the compactification radii under the $\mathrm{SL}(2,\mathbb{Z})_N$ transformations.

Keywords: D-branes, M(atrix) Theories, String Duality.

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Contents

1. Introduction

Recently noncommutativity of space-time coordinates has emerged in a number of occasions in string theory. After the discovery of the significance of D-branes as carriers of RR charge in string theory [1], it was observed that embedding coordinates of D-branes are in fact noncommutative [2]. The reason for this surprising result is that, dynamics of N coincident D_p-branes in low energy regime can be shown to be described by a supersymmetric Yang-Mills (SYM) SU(N) gauge theory in $p+1$ dimensions, obtained by dimensional reduction of 10 dimensional $N=1$ SYM theory. Through this dimensional reduction, the components of gauge field corresponding to transverse direction to the brane behave as scalar fields of the $(p+1)$ dimensional gauge theory. These scalars are the transverse coordinates of the D-brane, and hence result in the noncommutativity of space coordinates.

These noncommutative coordinates in the case of 0-branes are elevated to the dynamical variables of Matrix-model which is conjectured to describe the strong coupling limit of string theory, or M-theory, in the infinite momentum frame [3].

In Matrix-model the noncommutativity of matrices, and therefore the coordinates, become significant in substringy scales, as expected from general quantum gravitational considerations.

Another type of space-time noncommutativity has been recently observed in M-theory which is naively different from the above noncommutativity. It arose from the application of the non-commutative geometry (NCG) techniques pioneered by A. Connes to the Matrix-model compactifications [4].

According to Matrix-model conjecture, each momentum sector of the discrete light cone quantization (DLCQ) of M-theory is described by a maximally supersymmetric Matrix-model (or SYM), with the light cone momentum identified with the rank of gauge group. This conjecture has passed many consistency checks; for a review of Matrix model see [6, 7, 8]. To be a formulation of M-theory, Matrix model must describe string theory when compactified on a circle. Moreover, one should consider further compactifications of Matrix model, and check the conjectured U-duality groups of M-theory in various compactifications. But Matrix model compactifications involve complicated operations and it is not at all clear how to obtain them in general. A class of toroidal compactifications were constructed in early stages of Matrix model development, which relied on a certain commutative subalgebra of matrices [8, 9]. In a certain sense this subalgebra is an equivalent description of the manifold of torus on which compactification is performed.

It was observed by Connes, Douglas and Schwarz (CDS) that generalizing this same algebraic description of the manifold of compactification, in the spirit of NCG, to a noncommutative torus, it is possible to arrive at a different compactification of Matrix-model and different physical consequences, which is equivalent to adding a constant 3-form background in the 11 dimensional supergravity. A major result is that the SYM theory of commutative torus compactification now becomes a "deformed" SYM theory, with important non-local interactions introduced [10, 11]. Soon after, it was observed by Douglas and Hull [10] that deformed SYM theory and, therefore indirectly, the noncommutative torus (NCT) compactification is a natural consequence of certain D-brane configurations in string theory.

Subsequently compactification on more complicated spaces were studied in [12] and various properties of the deformed SYM theory and their relation to string theory were considered [13, 14, 15].

It is then clear that there is a close connection between non-zero constant background Kalb-Ramond anti-symmetric field ($B_{\mu\nu}$) and deformation of the torus of compactification of Matrix model and the non-locality of the resultant deformed field theory on torus. Yet it was not obvious how the turning the background B field on, causes the coordinates to become noncommutative and how this noncommutativity differs from that of coincident D-branes.

In this article[1] we propose an explicit construction of this noncommutativity and compare it with the noncommutativity due to coincident D-branes. We will show that a string theory membrane wrapped around T^2 in the presence of background B field, manifests noncommutative coordinates as a simple consequence of canonical commutation relations. We then show that applying T-duality and using the DVV string matrix theory [17] relation of Matrix model to string theory, results in Matrix model compactification on a deformed torus.

[1]A preliminary version of this work was presented in PASCOS98 [16]

The plan of the paper is as following. In section 2, we review the CDS construction [4]. Section 3, contains the explicit noncommutative coordinate construction of the wrapped membrane; and section 4, is devoted to the mass spectrum and its symmetries. In section 5, we will compare our string theory results with the other works in this subject and discuss the role of DBI action.

2. Compactification on a noncommutative torus

Matrix model describes M-theory in the infinite momentum frame. The dynamical variables are $N \times N$ matrices which are function of time, and N is taken to infinity. Matrix model is described by the supersymmetric action,

$$I = \frac{1}{2g\sqrt{\alpha'}} \int d\tau \mathrm{Tr} \Big\{ \dot{X}_a \dot{X}_a + \frac{1}{(2\pi\alpha')^2} \sum_{a<b} \left[X^a, X^b\right]^2 + \\ + \frac{i}{2\pi\alpha'} \Psi^T \dot{\Psi} - \frac{1}{(2\pi\alpha')^2} \Psi^T \Gamma_a [X^a, \Psi] \Big\}, \tag{2.1}$$

where $X^a, a = 1, \ldots, 9$ are bosonic hermitian matrices and Ψ are 16 component spinors. Γ^a are SO(9) Dirac matrices. Classical time independent solutions have commuting X^a, therefore simultaneously diagonalizable, corresponding to the classical coordinates of N 0-branes. In general off-diagonal elements of X^a correspond to substringy noncommutative structure of M-theory. This theory as a candidate for M-theory has passed a number of tests.

Compactification of coordinate X of Matrix model on a space-like circle of radius R has been shown [9] to require existence of the matrix U with the property

$$\begin{aligned} UXU^{-1} &= X + R\,, \\ UX^aU^{-1} &= X^a\,, \qquad X^a \neq X\,, \\ U\Psi U^{-1} &= \Psi\,. \end{aligned} \tag{2.2}$$

It was then shown that the solution of these equations can be written in terms of a covariant derivative

$$\begin{aligned} X &= i\frac{\partial}{\partial\sigma} + A\,, \\ U &= e^{i\sigma R}\,, \end{aligned} \tag{2.3}$$

and when substituted in the original action I, it becomes that of a $(1+1)$ dimensional SYM on the dual circle. This $(1+1)$ dimensional space is parametrized by (σ, τ), and the coupling constant of this theory is $g_{YM}^2 \sim 1/R$.

It was then shown that this $(1+1)$ dimensional SYM theory is identical to the IIA string theory for string scales, as expected [17]. The DVV map which relates the matrices to the strings plays an important role in our description of the noncommutativity in Matrix theory. The coupling constant of the string varies as inverse of the g^2_{YM}, $g_s = 1/\alpha' g^2_{YM}$, the dimension of the matrices, N, is carried into the light cone energy p^+ of strings and the eigenvalues of the matrices correspond to N free strings in the limit of vanishing g_s.

Compactification on a 2 torus is similarly accomplished by solving the equations

$$
\begin{aligned}
U_1 X_1 U_1^{-1} &= X_1 + R_1\,, \\
U_2 X_2 U_2^{-1} &= X_2 + R_2\,, \\
U X^a U^{-1} &= X^a\,, \qquad a \neq 1,2\,, \\
U \Psi U^{-1} &= \Psi\,,
\end{aligned}
\tag{2.4}
$$

But now consistency between these equations requires:

$$
U_1 U_2 = e^{i\theta} U_2 U_1\,, \tag{2.5}
$$

for some real number θ; where for the usual commutative torus, $\theta = 0$. We will later see that in fact a *rational* θ will also give a commutative torus. Again it is easily seen that for $\theta = 0$,

$$
\begin{aligned}
X_i &= i\frac{\partial}{\partial \sigma_i} + A_i\,, \qquad i = 1,2\,, \\
U_i &= e^{i\sigma_i R_i}\,.
\end{aligned}
\tag{2.6}
$$

is a solution of eqs. (2.4) and (2.5) and its insertion in the action results in the 2+1 dimensional SYM on the dual torus. Here σ_i parameterize the dual two-torus.

Connes, Douglas and Schwarz [4] observed that in eq. (2.5), θ can be taken different from zero and it corresponds to compactification on a noncommutative torus (NCT) and the resulting gauge theory is the SYM with the commutator of the gauge fields replaced by the Moyal bracket. The central idea of NCG is, starting from the equivalence of a manifold with the c^* algebra of functions over that manifold, to generalize to a noncommutative c^* algebra [18]. Thus, the algebra generated by the commuting matrices U_1 and U_2 in the case of usual T^2, is generalized [12, 18, 19] to the algebra generated by U_1 and U_2 satisfying the relation (2.5), which now defines a "noncommutative" torus, T^2_θ. The solutions of (2.5) are then,

$$
X_i = -iR_i \partial_i + A_i\,, \tag{2.7}
$$

where A_i now are functions of $\tilde{U}_i$, with $\tilde{U}_i$ satisfying

$$
\begin{aligned}
\tilde{U}_1 \tilde{U}_2 &= e^{-i\theta} \tilde{U}_2 \tilde{U}_1\,, \qquad U_i \tilde{U}_j = \tilde{U}_j U_i\,, \\
[\partial_i, \tilde{U}_j] &= i\delta_{ij} \tilde{U}_j\,, \qquad i,j = 1,2\,.
\end{aligned}
\tag{2.8}
$$

Substituting them in the action, we get the SYM theory on the NCT dual to the original one, with the essential modification being, the replacement of commutators of gauge fields by the Moyal bracket,

$$\{A, B\} = A * B - B * A\,, \tag{2.9}$$
$$A * B(\sigma) = e^{-i\theta(\partial_1'\partial_2'' - \partial_2'\partial_1'')} A(\sigma')B(\sigma'')|_{\sigma'=\sigma''=\sigma}\,, \tag{2.10}$$

with $\sigma = (\sigma_1, \sigma_2)$. Moyal bracket introduces non-locality into the theory. This theory obviously suffers from lack of Lorentz invariance in the substringy scales; however, has better convergence properties compared with the ordinary SYM theory [10, 13].

An important test of the conjecture that the compactification of the Matrix model in the presence of non-zero 3-form background field is equivalent to the SYM theory on a NCT, is comparison of the mass spectra of the two theories. This comparison, in the case of BPS states, was carried out in CDS, by giving the BPS spectrum of SYM on NCT with the low energy BPS states in 11 dimensional supergravity in the presence of the three form C in the light cone direction.

Ho [19] calculated the same BPS spectrum for the Matrix theory compactified on NCT, with certain modifications, i.e. to take into account the longitudinal and transverse membrane winding modes, which is in fact equivalent, as we will show in section 4, to turning on the $B_{\mu\nu}$ background field. He obtained the energy of BPS states,[2]

$$E = \frac{R}{n - m\theta}\left\{\frac{1}{2}\left(\frac{n_i - m_i\theta}{R_i}\right)^2 + \frac{V^2}{2}[m + (n - m\theta)\gamma]^2 \right. \\ \left. + 2\pi\sqrt{(R_1w_1)^2 + (R_2w_2)^2}\right\}, \tag{2.11}$$

where $V = (2\pi)^2 R_1 R_2$ and n_i/R_i are KK momenta conjugate to X_i; $m_i = \epsilon_{ij} m_{j-}$, with m_{i-} winding number of the longitudinal membrane along X_i and X_- direction; R the compactification radius along the X_- direction and w_i are the momenta of BPS states due to the transverse coordinates and are constrained by:

$$w_i = \epsilon_{ij}(nm_j - mn_j)\,. \tag{2.12}$$

Moreover n is the dimension of matrices (number of 0-branes), m the winding number of the membrane around torus and θ is the deformation parameter of the torus. It is then noted that this spectrum is the same as that obtained from the SYM theory on NCT, where $m = \epsilon_{ij} m_{ij}$ [4, 19]. The term involving γ is essentially put in an ad hoc manner and is mainly needed for the $\mathrm{SL}(2, \mathbb{Z})_N$ symmetry below. This term

[2] We would like to thank the referee for pointing out that, although Ho obtained this result from a modified Matrix model action which had no justification, the spectrum is still valid.

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corresponds to an arbitrariness in the mass formula of CDS. We will summarize and compare these results with CDS's and with ours at the end of section 4.

An important property of the mass spectrum (2.12) is its $\mathrm{SL}(2,\mathbb{Z})_N$ invariance generated by

$$
\begin{aligned}
&\theta \to -1/\theta \qquad && m \to n\,, \qquad && n \to -m \\
& && m_i \to n_i\,, && n_i \to -m_i \\
&\gamma \to -\theta(\theta\gamma+1) && R_i \to \theta^{-2/3} R_i\,, && R \to \theta^{-1/3} R
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
&\theta \to \theta+1 \qquad && n \to n+m\,, \qquad && m \to m \\
& && n_i \to n_i + m_i\,, && m_i \to m_i\,.
\end{aligned}
\tag{2.14}
$$

This invariance is to be expected on the basis of the NCG considerations. It is the $\mathrm{SL}(2,\mathbb{Z})$ invariance of the c^*-algebra defining the NCT [4].

We note that from noncommutative geometric arguments, CDS observed that the commutator of the NCT coordinates should satisfy

$$
[X^1, X^2] = 2\pi i R_1 R_2 \frac{m}{n - m\theta}\,. \tag{2.15}
$$

We will later obtain this relation from string theory.

3. Noncommutativity from string theory

In this section we trace the noncommutativity of the Matrix model compactification in string theory formulated in the presence of the antisymmetric background field. The noncommutativity appears in string theory when we consider D-branes living in the $B_{\mu\nu}$ background.

We begin with the action of Fundamental strings ending on a D-membrane in the background of the antisymmetric field, $B_{\mu\nu}$ [20]:

$$
S = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \Big[\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu g^{ab} + \epsilon^{ab} B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\Big] + \frac{1}{2\pi\alpha'}\oint_{\partial\Sigma} d\tau A_i \partial_\tau \zeta^i\,, \tag{3.1}
$$

where A_i, $i = 0,1,2$ is the $U(1)$ gauge field living on the D-membrane and ζ^i its internal coordinates. The action is invariant under the combined gauge transformation [2]

$$
\begin{aligned}
B_{\mu\nu} &\to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\
A_\mu &\to A_\mu - \Lambda_\mu\,.
\end{aligned}
\tag{3.2}
$$

The gauge invariant field strength is then

$$
\mathcal{F}_{\mu\nu} = B_{\mu\nu} - F_{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_{[\mu} A_{\nu]}\,. \tag{3.3}
$$

Variation of the action S, leads to the following mixed boundary conditions

$$\begin{cases} \partial_\sigma X^0 = 0 \\ \partial_\sigma X^1 + \mathcal{F}\partial_\tau X^2 = 0 \\ \partial_\sigma X^2 - \mathcal{F}\partial_\tau X^1 = 0\,, & \mathcal{F} = \mathcal{F}_{12} \\ \partial_\tau X^a = 0\,, & a = 3,\ldots,9\,. \end{cases} \tag{3.4}$$

Imposing the canonical commutation relation on X^i and its conjugate momenta P^i, $i = 1, 2$:

$$P^1 = \partial_\tau X^1 - \mathcal{F}\partial_\sigma X^2\,, \qquad P^2 = \partial_\tau X^2 + \mathcal{F}\partial_\sigma X^1\,. \tag{3.5}$$

$$[X^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma')\,. \tag{3.6}$$

Leads to the non-trivial relation[3]

$$[X^1(\sigma,\tau), X^2(\sigma',\tau)] = 2\pi i\mathcal{F}\theta(\sigma-\sigma'). \tag{3.7}$$

Mode expansions for X^1 and X^2 consistent with our boundary conditions, are

$$\begin{cases} X^1 = x_0^1 + (p^1\tau - \mathcal{F}p^2\sigma) + \sum_{n\neq 0} \frac{e^{-in\tau}}{n}\Big(ia_n^1 \cos n\sigma + \mathcal{F}a_n^2 \sin n\sigma\Big) \\ X^2 = x_0^2 + (p^2\tau + \mathcal{F}p^1\sigma) + \sum_{n\neq 0} \frac{e^{-in\tau}}{n}\Big(ia_n^2 \cos n\sigma - \mathcal{F}a_n^1 \sin n\sigma\Big)\,, \end{cases} \tag{3.8}$$

from which the center of mass coordinates

$$x^i = \frac{1}{\pi}\int X^i(\sigma,\tau)\; d\sigma \tag{3.9}$$

satisfies

$$[x^1, x^2] = \pi i\mathcal{F}\,. \tag{3.10}$$

We claim that this noncommutativity of space coordinates is at the root of the geometric noncommutativity which appears in the compactification of Matrix model on a torus in the background 3-form field, as described in section 2. To show this, we will map the coordinate X^i to the Matrix model variables via the string matrix model of DVV, and use it to construct the NCT compactification discussed in the previous section.

But, first we would like to elaborate on the connection between the noncommutativity of (3.10) and the noncommutativity which appears in the transverse coordinates of several coincident D-branes. The point is that it has been shown that D-membranes with a non-zero U(1) gauge field in the background contain a distribution of 0-branes proportional to F [16, 21, 22]. In previous works only the zero $B_{\mu\nu}$ case were considered. These 0-branes as described by the Matrix model, live

[3] A non-zero $\mathcal{F}_{0i}$, will not give any noncommutativity between X^0 and X_i. This is the effect of the world sheet metric signature.

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on a torus with a D-membrane wrapped on, with noncommutative coordinates X^1 and X^2,

$$[X^1, X^2] = if\,. \tag{3.11}$$

The proportionality constant f is given by the U(1) gauge field strength $\mathcal{F}$ [22].

It is quite remarkable that the elaborate mechanism which produces the original noncommutativity in the description of D-branes, first discovered by Witten, and leads through a set of subtle arguments to the particular form of the commutation relation in (3.11), should be simply derived from the string action (3.1) in the presence of the $\mathcal{F}$ and mixed boundary conditions for zero B field background.

We will shortly see that the noncommutativity (3.10), under certain circumstances, leads to the noncommutative torus compactification of CDS. To see this we compactify the X^i direction and wrap the 2-brane around the 2-torus and use the center of mass coordinates x^i and their conjugate momenta to construct the generators of the c^* algebra of the noncommutative torus; proving that the compactification, in the presence of $U(1)$ field strength, for D-membrane requires a NCT. Thus we demand to solve the compactification equations for the membrane coordinates x^i

$$\begin{aligned}
U_1 x^1 U_1^{-1} &= x^1 + R_1\,,\\
U_2 x^2 U_2^{-1} &= x^2 + R_2\,,\\
U_i x^j U_i^{-1} &= x^j\,, \qquad i \neq j = 1,2\,.
\end{aligned} \tag{3.12}$$

A solution to these equations is:

$$\begin{aligned}
U_1 &= exp\left\{-iR_1\left[a\left(p_1 - \frac{x^2}{\pi\mathcal{F}}\right) - \frac{x^2}{\pi\mathcal{F}}\right]\right\},\\
U_2 &= exp\left\{-iR_2\left[a\left(p_2 + \frac{x^1}{\mathcal{F}}\right) + \frac{x^1}{\mathcal{F}}\right]\right\},
\end{aligned} \tag{3.13}$$

with $a^2 = 1 + \frac{\pi^2\mathcal{F}^2}{R_1 R_2}$. The above relations leads to

$$U_1 U_2 = e^{i\pi\mathcal{F}} U_2 U_1\,. \tag{3.14}$$

This result is similar to the Matrix theory compactification on the NCT formulated by CDS, described previously. It was argued there that, the noncommutativity of the torus is related to the non-vanishing of 3-form of M-theory, which in the string theory reduces to the antisymmetric NSNS 2-form field, $B_{\mu\nu}$. In our case noncommutativity of the torus on which the D-membrane of string theory is compactified, is a direct result of the non-vanishing B field. In fact using the Matrix model formulation of string theory [17], it is straightforward to obtain CDS results. In the string matrix model of DVV, the matrices $X^\mu(\tau)$ of Matrix theory, upon compactification of a space-like dimension, say X^9, become matrices $X^\mu(\tau,\sigma)$, receiving a σ dependence, and satisfying the Green-Schwarz action of the string theory in the light-cone frame;

with their noncommutativity reflecting the added D-brane structure in string theory. Using the relation between conventional string theory and string matrix model, we map our noncommuting membrane coordinates X^i, $i = 1, 2$, and the noncommutative torus generators U_i, to the Matrix theory and obtain (2.4) and (2.5) for compactification of Matrix theory on the NCT.

The noncommutativity of the c^* algebra (3.14) and (2.5) of the NCT is similar to, but distinct from, the noncommutativity of the coordinates as in (3.10) and as it appears in Matrix theory and bound states of D-branes. The similarities are obvious, but the differences are subtle. In fact it is possible to see that when $\mathcal{F}$ is quantized to a rational number, by an $\mathrm{SL}(2, \mathbb{Z})$ transformation, we can make the U_1 and U_2 commute, i.e. we can make the torus *commutative*, while the coordinates are *noncommutative*. Thus for *irrational* parameter θ, we are dealing with a new form of noncommutativity not encountered in ordinary Matrix theory or in the context of D-brane bound state.

4. The BPS spectrum

To complete our string theoretic description of the CDS formulation, i.e. the SYM theory on NCT, we find the BPS spectrum of a system of (D2-D0)-brane bound state. To have an intuitive picture, it is convenient to consider the T-dual version of the mixed brane discussed earlier in section 3. The advantage of T-duality is that in T-dual picture we only deal with commutative coordinates and commutative torus, where we are able to calculate the related spectrum by the usual string theory methods.

Applying T-duality in an arbitrary direction, say X^2, (3.4) results in

$$\begin{cases} \partial_\sigma X^0 = 0\,, \\ \partial_\sigma (X^1 + \mathcal{F} X^2) = 0\,, \\ \partial_\tau (X^2 - \mathcal{F} X^1) = 0\,, \\ \partial_\tau X^a = 0\,, \qquad a = 3, \dots, 9\,, \end{cases} \tag{4.1}$$

describing a tilted D-string which makes an angle ϕ with the duality direction, X^2:

$$\cot\phi = \mathcal{F}\,.$$

Thus we consider a D-string winding around a cycle of a torus defined by:

$$\tau = \frac{R_2}{R_1} e^{i\alpha} = \tau_1 + i\tau_2\,, \qquad \rho = iR_1 R_2 \sin\alpha + b = i\rho_2 + b\,, \tag{4.2}$$

where $b = BR_1R_2 \sin\alpha$ is the flux of the B field on the torus. The D-string is located at an angle ϕ with the R_1 direction such that it winds n times around R_1 and m times around R_2. Hence

$$\cot\phi = \frac{n}{m\tau_2} + \cot\alpha\,. \tag{4.3}$$

The greatest common divisor of m and n is the number of times D-string winds around that cycle specified with the angle ϕ.

The BPS spectrum of this tilted D-string system gets contributions from both the open strings attached to the D-string and the D-string itself. In order to consider the most general case, we assume a moving D-string which also has a non-zero electric field living on it.

4.1 Open strings contributions

As in [23, 24], the brane velocity and its electric field will not affect the open strings spectrum. So it is sufficient to consider the open strings satisfying (4.1). These open strings have mode expansions [16]:

$$\begin{cases} X^i = x_0^i + p^i\tau + L^i\sigma + \text{Oscil.}\,, & i = 1, 2 \\ X^0 = x_0^0 + p^0\tau + \text{Oscil.}\,, & \\ X^a = x_0^a + \text{Oscil.}\,, & a = 3, \ldots, 9 \end{cases} \tag{4.4}$$

where p^i and L^i, in usual complex notation, are:

$$p = \; r_1 \frac{n + m\tau}{|n + m\tau|^2} \sqrt{\frac{\tau_2}{\rho_2}}\,, \qquad r_1 \in \mathbb{Z}\,. \tag{4.5}$$

$$L = \; q_1 \frac{\rho(n + m\tau)}{|n + m\tau|^2} \sqrt{\frac{\tau_2}{\rho_2}}\,, \qquad q_1 \in \mathbb{Z}\,. \tag{4.6}$$

As we can see, p is parallel to the D-string. We should note that, in the case of non-zero B field, L is no longer perpendicular to D-string. Moreover $|L|$ is an integer multiple of a minimum length. This is the length of a string stretched between two consecutive cycles of the wound D-string [25].

Mass of the open string defined by (4.4) is

$$M^2 = |p + L|^2 + \mathcal{N} = \frac{\tau_2}{|n + m\tau|^2} \frac{|r_1 + q_1\rho|^2}{\rho_2} + \mathcal{N}\,, \tag{4.7}$$

where $\mathcal{N}$ is the contribution of the oscillatory modes. As it is seen, (4.7) is manifestly invariant under both SL$(2,\mathbb{Z})$'s of the torus acting on ρ and τ. To find the contribution of the open string to BPS spectrum of the membrane on T_θ^2, we apply T-duality in R_1 direction,

$$R_1 \to \frac{1}{R_1}\,, \qquad \text{or equivalently} \qquad \tau \leftrightarrow \rho\,,$$

and obtain the spectrum of the open string compactified on NCT,

$$M^2 = \frac{\rho_2}{|n + m\rho|^2} \frac{|r_1 + q_1\tau|^2}{\tau_2} + \mathcal{N}\,, \tag{4.8}$$

with the U(1) gauge field,

$$\mathcal{F}^{-1} = \frac{n}{m\rho_2} + \cot\alpha\,. \tag{4.9}$$

The above relation shows that $\mathcal{F}$ takes contributions from both the torus ($\cot\alpha$) and the D-string tilt ($n/m\rho_2$). Rewriting (3.7) for T-dual of open string mode expansions (4.4), we get

$$[X^1, X^2] = 2\pi i\rho_2\left(\frac{n}{m} + \rho_2\cot\alpha\right)^{-1} = 2\pi i\rho_2\left(\frac{n}{m} - b\right)^{-1}. \tag{4.10}$$

The above equation is the same as the commutation relation between coordinates of a deformed torus in NCG, (2.15), where θ is substituted for the b field, and the b itself, through T-duality, is related to the angle of torus.

Note that a rational b field will not give a NCT as shown in [4]. In our string theoretic description this is easily seen from the T-dual version, where, by means of a $\mathrm{SL}(2,\mathbb{Z})$ transformation we can transform such tori to an orthogonal torus, giving a zero b field after T-duality. Hence, only the irrational part of b field can not be removed by $\mathrm{SL}(2,\mathbb{Z})_N$ transformations.

4.2 The D-string contribution

Now we consider the most general case of a D-string on a torus, i.e. a moving D-string which has a non-zero electric field. To handle this problem, we use the DBI action which gives the dynamics of D-strings. It has been shown in [23], for a D-string with an electric field, [24] for the moving brane case, and [21], for a D-brane with a magnetic field, that the mass of a D-brane, calculated from DBI or string theory, coincide.

Consider the DBI action for the above tilted D-string moving with velocity v normal to the D-string and the gauge field F parallel to it in a non-zero B_{12} background.[4] Here we assume that the RR scalar is zero:

$$S_{D-string} = -\frac{1}{g_s}\int d^2\sigma\sqrt{\det(\eta_{ab} + \mathcal{F}_{ab})}\,. \tag{4.11}$$

For the D-string discussed above, we have

$$\eta_{ab} = \begin{pmatrix} 1-v^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{F}_{ab} = \begin{pmatrix} 0 & Bv+F \\ Bv+F & 0 \end{pmatrix}. \tag{4.12}$$

Inserting (4.12) in (4.11), we get

$$S_{D-string} = -\frac{1}{g_s}\int d^2\sigma\sqrt{1 - v^2 - (F+Bv)^2}\,. \tag{4.13}$$

[4]Because the DBI action for a D_p-brane has SO(p,1) symmetry, only the velocities normal to brane are relevant.

To calculate the mass spectrum of D-string, we consider the Hamiltonian for (4.13),

$$H = \frac{1}{g_s}\int d\sigma\sqrt{1+[(P-\Pi B)^2+\Pi^2]g_s^2}\,, \tag{4.14}$$

where P, Π are the conjugate momenta of collective coordinate of D-string and the electric gauge field, respectively:

$$P = \frac{\partial L}{\partial v} = \frac{1}{g_s}\frac{v+B(F+Bv)}{\sqrt{1-v^2-(F+Bv)^2}}\,. \tag{4.15}$$

$$\Pi = \frac{\partial L}{\partial F} = \frac{1}{g_s}\frac{F+Bv}{\sqrt{1-v^2-(F+Bv)^2}}\,. \tag{4.16}$$

P, Π defined on the dual torus, should be quantized [2]:

$$P = \frac{r_2}{\rho_2}\frac{1}{|n+m\tau|}\,, \qquad \Pi = \frac{q_2}{|n+m\tau|}\,. \tag{4.17}$$

Plugging (4.17) into (4.14), we have

$$\alpha' M^2 = \frac{|n+m\tau|^2\rho_2}{\alpha' g_s^2\tau_2} + \alpha'\frac{|r_2+\rho q_2|^2}{\rho_2\tau_2}\,. \tag{4.18}$$

Applying T-duality on the (4.18), we obtain a (D2-D0)-brane bound state in the presence of a non-vanishing B field. Also we have turned on the electric fields living on the D2-branes world-volume. These electric fields are given by T-dual version of (4.17).

Hence the mass spectrum of the membrane discussed above is[5]

$$\alpha' M^2_{membrane} = \frac{|n+m\rho|^2\tau_2}{\alpha' {g'_s}^2\rho_2} + \alpha'\frac{|r_2+\tau q_2|^2}{\rho_2\tau_2}\,. \tag{4.19}$$

The $\mathrm{SL}(2,\mathbb{Z})_N$ invariance, acting on ρ, is manifestly seen from the above equation.

4.3 The full spectrum

As shown in [26], the open strings discussed earlier and the D-string form a *marginal* bound state, i.e. from the brane gauge theory point of view, the open strings are electrically charged particles with non-vanishing Higgs fields. So to find the full BPS spectrum, we should add the masses and not their square:

$$\mathcal{M} = M_{membrane} + M_{open\ st.}\,. \tag{4.20}$$

$$\mathcal{M} = \sqrt{\frac{\tau_2}{\rho_2}}\frac{|n+m\rho|}{g'_s}\left(1+{g'_s}^2\frac{|r_2+q_2\tau|^2}{\tau_2}\frac{\rho_2}{|n+m\rho|^2}\right)^{1/2} + \frac{|r_1+q_1\tau|}{|n+m\rho|}\sqrt{\frac{\rho_2}{\tau_2}}\,. \tag{4.21}$$

[5]We should note that under T-duality the string coupling constant behaves as $g_s \to g'_s = g_s\sqrt{\tau_2/\rho_2}$.

The above spectrum is manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant. In the usual notation of T^2_θ [4], the $\mathrm{SL}(2,\mathbb{Z})_N$ acting on ρ is non-classical.

To compare our results with [4] or [19], we should find the zero volume and $g_s \to 0$ limits. These limits are necessary for comparison, as the Matrix theory used in [4, 19], is described by the 0-brane dynamics at small couplings, compactified on a light-like direction.

In the absence of a B field background, applying the $\mathrm{SL}(2,\mathbb{Z})_N$ transformation $\rho \to -1/\rho$, we can go to the large volume limit, sending the number of D0-branes to infinity, reproducing the results of large N Matrix model; but if the B field is non-zero the above transformation,

$$iV + B \to \frac{iV - B}{V^2 + B^2}\,, \tag{4.22}$$

does not allow V to go to infinity. Hence the number of D0-branes distributed on D-membrane remains finite. In the zero volume limit, which is not altered by the above $\mathrm{SL}(2,\mathbb{Z})_N$ transformation, we end up with a finite number of D0-branes.

In these limits up to V^2 and g_s

$$|n + m\rho| = |(n - m\theta) + iVm| = |n - m\theta| + \frac{m^2V^2}{2|n - m\theta|} + O(V^3)\,,$$

and hence (4.21) reads as

$$l_s\mathcal{M} = \frac{|n - m\theta|}{g_s} + \frac{1}{2g_s}\frac{m^2V^2}{|n - m\theta|} + \frac{g_s}{2|n - m\theta|}\frac{|r_2 + q_2\tau|^2}{\tau_2} + \frac{|r_1 + q_1\tau|}{|n - m\theta|}\sqrt{\frac{V}{\tau_2}}\,. \tag{4.23}$$

The first term is due to the D2-brane itself;[6] the second term is due to the magnetic flux or the D0-brane contribution; the third term is the contribution of electric fields; and the last term belongs to open strings.

We observe that this mass spectrum is equivalent to the spectrum given by Ho [19] (eq.(2.11)). Comparing the spectrum (4.22) with the BPS spectrum of [4] and [19] we can construct the following correspondence table (table 1).

As we see the interpretation of $p_- = (n - m\theta)/R$ in our case is the mass of the tilted D-string which is closely related to the DVV's string matrix theory in the noncommutative case.

The $\mathrm{SL}(2,\mathbb{Z})_N$ symmetry generators of (4.21) are

$$\rho \to \rho + 1\,, \qquad \rho \to -1/\rho\,,$$

that in the zero volume limit ($\rho_2 = 0$) become

$$\theta \to \theta + 1\,, \qquad \theta \to -1/\theta\,. \tag{4.24}$$

[6] If we consider the full DBI action, which also has a constant term added due to D2-brane RR charge, this term will be removed.

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SYM/T^2_θ	$D=11$ $SUGRA/S_-$	Matrix theory on T^2_θ	String theory (this work)
$P^i = \int T^{0i}$	nm^{i-}	$n\epsilon^{ij}m_{j-} = nm^i$	a combination
e^i	e^i	n^i	of $r_2,\ q_2$
p'^i	w^i	w^i	$r_1,\ q_1$
p	$n\ (p_- = \frac{n-m\theta}{R})$	n	n
q	m^{ij}	$m = m^{ij}\epsilon^{ij}$	m
θ	RC_{-ij}	θ	$b = B$ field flux

Table 1: Comparison of parameters of [4, 19] with ours.

Under the above transformations (n, m) transforms as an $\mathrm{SL}(2,\mathbb{Z})_N$ vector. There is also an $\mathrm{SL}(2,\mathbb{Z})_C$ symmetry of mass spectrum (4.22), acting on the τ, under which the (r_i, q_i) behave as $\mathrm{SL}(2,\mathbb{Z})$ vectors.

Invariance of the mass spectrum, (4.22), under $\theta \to -1/\theta$, implies that

$$g_s \to g'_s = g_s\theta^{-1}\,. \tag{4.25}$$

Moreover the imaginary part of $\rho \to -1/\rho$, tells us that the volume of the torus in the zero volume limit, in the string theory units, transforms as:

$$V \to V' = V\theta^{-2}\,. \tag{4.26}$$

Putting these relations together, and remembering the relation of 10 dimensional units and 11 dinemsional parameters, $l_p^3 = l_s^3 g_s$ and $l_s g_s = R$, and assuming l_p invariance under θ transformations, we obtain:

$$\begin{aligned} R \to R' &= R\theta^{-2/3}\,, \\ R_i \to R'_i &= R_i\theta^{-2/3}\,, \\ l_s \to l'_s &= l_s\theta^{-1/3}\,. \end{aligned} \tag{4.27}$$

The above relations differ from the corresponding relation [4] or [19] (eq.(2.13)) and indicate an M-theoretic origin for the $\mathrm{SL}(2,\mathbb{Z})_N$. This is the effect of considering the whole DBI action and not, only its second order terms. The same result is also obtained from DBI by Ho [27].

5. Discussion

In this work we have studied more extensively the brane systems in a non-zero $B_{\mu\nu}$ background field, through the usual string theory methods, a problem also considered in [13, 14, 15].

In [14] D0-brane dynamics in B field background was considered, and shown that the B field modifies the D0-brane dynamics. As discussed there, the effects of such a background is to replace the Poisson bracket of fields by the noncommutative version, the Moyal bracket. The key idea there, is that the existence of background B field introduces a *phase factor* for the open strings attached to D0-branes, the phase factor being proportional to the background B field. These open strings, as discussed in [2, 3] carry the dynamical degrees of freedom of D-branes. Using this phase, they showed that this background will lead to a NC background in the related Matrix model. The same procedure in a slightly different point of view was considered in [15], supporting and clarifying the novel result in [4].

In this paper, extending the string theoretic ideas of [14, 15, 16], we have explicitly shown that the noncommutativity of brane coordinates come about naturally in the formulation of open strings in the background B field, as well as the noncommutativity of the torus.

To use the usual string theory methods, by means of T-duality, we replaced the torus defined by $\tau = \frac{R_2}{R_1}e^{i\alpha}$, by a torus with a B field, where the B flux is $R_1 R_2 \cos\alpha$. Hence in the T-dual version we dealt with a D-string wound around the cycle of the torus. Using the usual string theoretic arguments and also the DBI action for the corresponding D-string dynamics, we calculated the BPS spectrum of a system of (D2-D0)-brane bound state, in a B field background. As shown here, this brane system is described by a DBI action formulated on a noncommutative torus. In a remarkable paper [13] Li argued that SYM on T^2_θ will not fully describe the dynamics of D0-branes in a background B field, and DBI action becomes necessary. Our spectrum, in the small coupling and zero volume limit, reproduces the CDS results.

A novel feature in our work is that under the transformation $\rho \to -1/\rho$, R_i and the eleventh dimension compactification radius, R, transform in the same way; $R, R_i \to R', R'_i = R, R_i\theta^{-2/3}$, in contrast to the results of [4, 19] where SYM and Matrix model rather than the DBI action on a noncommutative torus, were used.

It is amusing that the dimension of $\mathcal{H}$ [4], $dim\mathcal{H} = |n - m\theta|$, is given by the length of the wrapped D-string. $dim\mathcal{H} = Tr\mathbf{1} = |n - m\theta|$ is a factor coming in front of the YM action on T^2_θ, which up to second order of m/n reproduces the higher power terms of DBI action in a B field background, as discussed in [13].

There are still a number of questions to be addressed: One is, how to realize the IIB $\mathrm{SL}(2,\mathbb{Z})$ in the Matrix theory on NCT. In other words, if we have both $B_{\mu\nu}$ and $\tilde{B_{\mu\nu}}$ (the RR two form), as background fields, how to incorporate this in the Matrix model on NCT. As indicated by Ho [28], the $\tilde{B_{\mu\nu}}$ is related to the g_{-i} component of the metric. So the problem of g_{-i} addressed by CDS, seems to be related to $\tilde{B}$ background.

Another interesting open question briefly studied in [11, 29] is in relation to the six dimensional theories. Six dimensional theories as theories living in the NS5-brane world volume, show up in the compactification of Matrix model on T^n, $n > 4$. These theories seem to be non-local field theories. On the other hand along the arguments of [10, 11], we know that considering B field background leads to non-local low energy field theories for open strings. The method we used here, i.e. applying T-duality to remove noncommutativity and replacing B field with the torus angle, may give new insight into the problem of NS5-branes (or six dimensional theories) in ordinary string theory.

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